Kinematics of a continuum

In kinematics we study the motion of a continuous medium (solid or fluid).

The study of motion provides various strain tensors used in the models of solid and fluid mechanics.

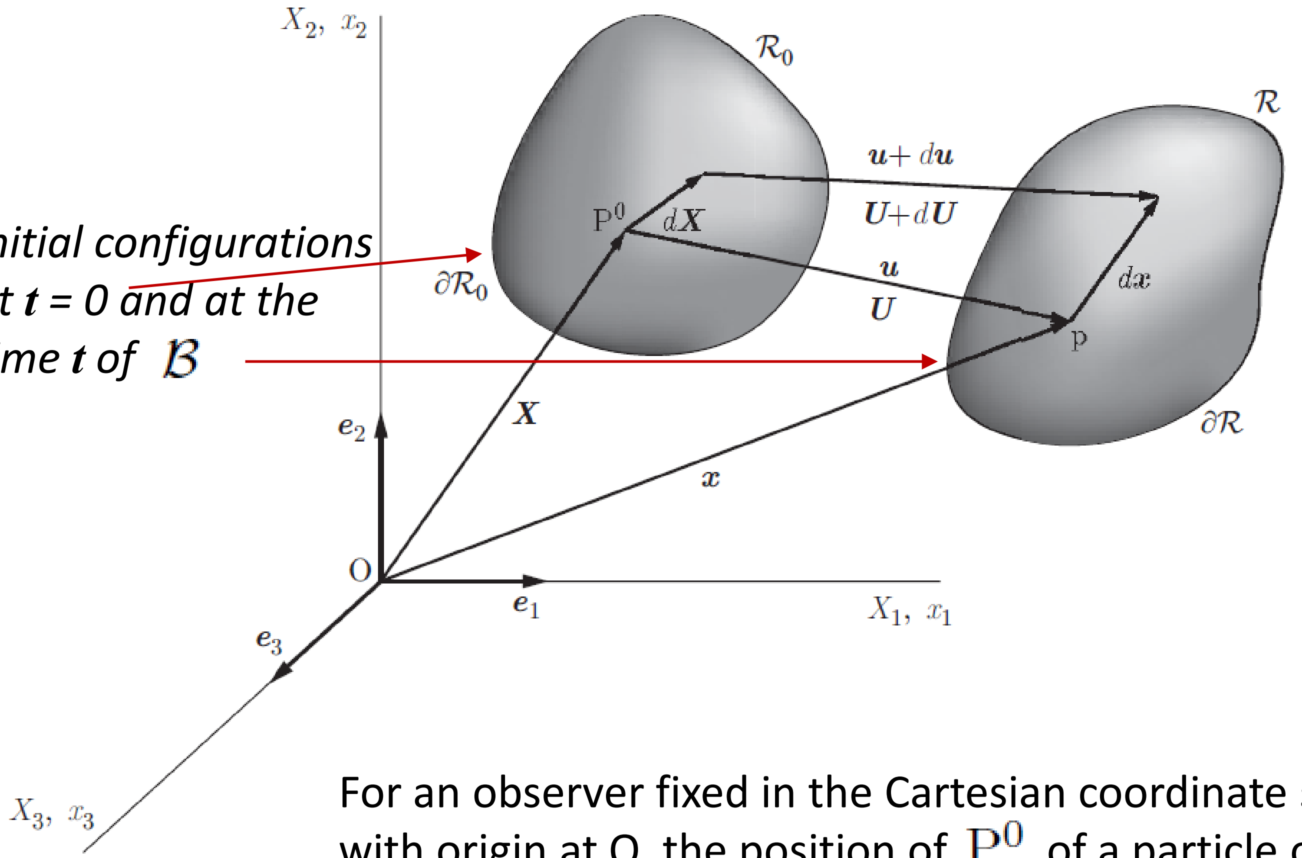
Strain tensors provide 'metrics' that can be used to measure changes in length during the motion of the solid.

From the book: Mechanics of Continuous Media: an Introduction, J Botsis and M Deville, PPUR 2018.
Solutions: https://www.epflpress.org/produit/908/9782889152810/mechanics-of-continuous-media

Continuum mechanics review: kinematics

CONFIGURATION AND MOTION

*Initial configurations at **t** = 0 and at the time **t** of* $\mathcal{B}$



For an observer fixed in the Cartesian coordinate system with origin at O, the position of P^0 of a particle of $\mathcal{B}$ at $t=0$ is represented by its **initial postion** vector $\boldsymbol{X}$ and its **current postion** p at time $t > 0$ by the current position vector $\boldsymbol{x}$.

The body $\mathcal{B}$ is defined as a set of particles or material points. These particles correspond to infinitesimal volumes around the points.

Volume occupied by all particles at time t gives the **configuration** of the body $\mathcal{R}_t$ or $\mathcal{R}$ with the initial configuration $\mathcal{R}_0$.

The **boundary** of the body is indicated by $\partial\mathcal{R}_0$ or $\partial\mathcal{R}$.

The **motion** of $\mathcal{B}$ is described by a vector function χ defined over time t that depends on $\boldsymbol{X}$.

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t)$$

➡ $\boldsymbol{X}$ is the initial position of a particle currently found at $\boldsymbol{x}$.

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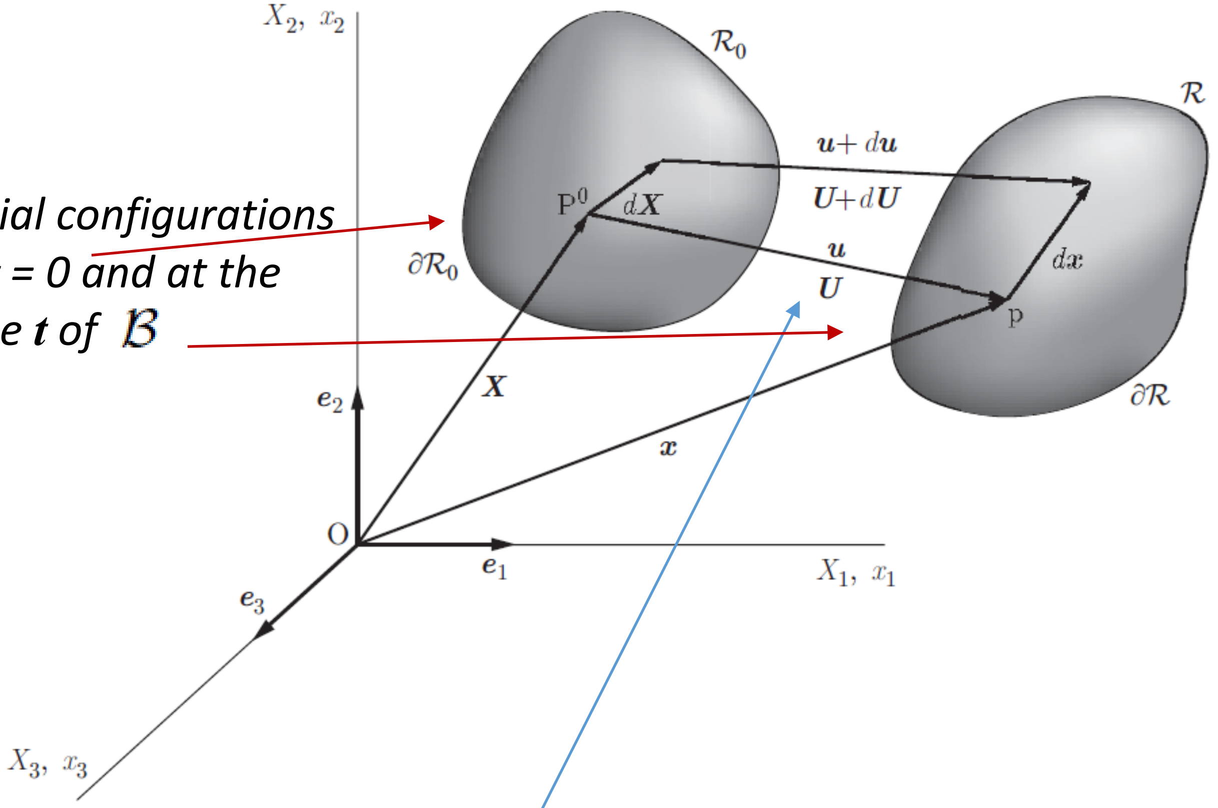
Configuration and motion



Initial configurations at ***t*** *= 0 and at the time* ***t*** *of* $\mathcal{B}$

By definition the vector displacement $\boldsymbol{u}$ is the vector difference:

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = \chi(\boldsymbol{X}, t) - \boldsymbol{X} = \boldsymbol{x} - \chi^{-1}(\boldsymbol{x}, t)$$

In the reference configuration ($t = 0$):

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t) \Rightarrow \boldsymbol{X} = \chi(\boldsymbol{X}, 0)$$

The motion is an one-to-one correspondence between the initial and current positions of the particles of $\mathcal{B}$.

The existence of the function $\chi : \mathcal{R}_0 \to \mathcal{R}$

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t)$$

and its inverse $\chi^{-1} : \mathcal{R} \to \mathcal{R}_0$

$$\boldsymbol{X} = \chi^{-1}(\boldsymbol{x}, t) \text{ with } \boldsymbol{X} = \chi^{-1}(\boldsymbol{X}, 0)$$

guarantees the integrality and unity of the body.

$$\Rightarrow \quad \chi(\chi^{-1}(\boldsymbol{x}, t), t) = \boldsymbol{x}$$
$$\chi^{-1}(\chi(\boldsymbol{X}, t), t) = \boldsymbol{X}$$

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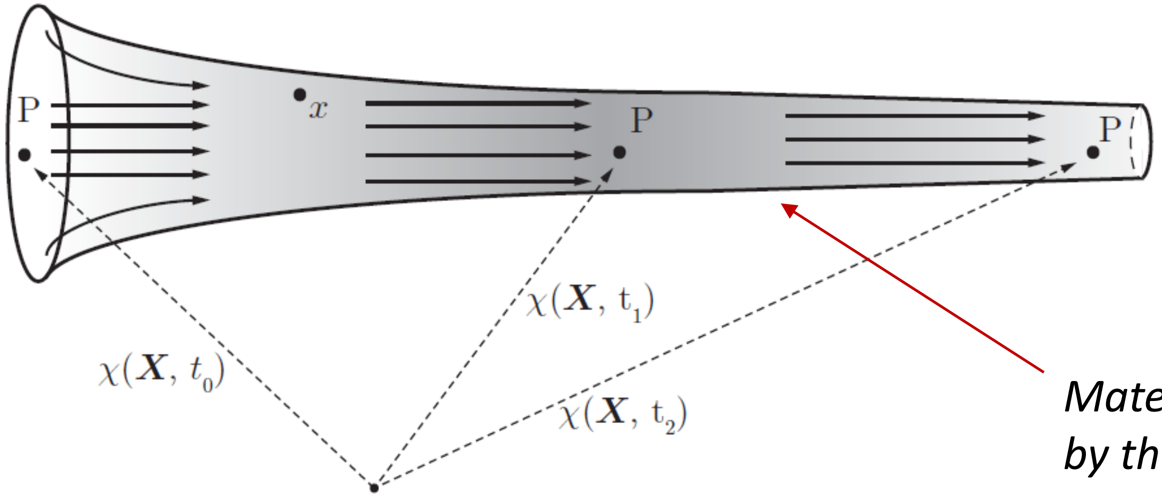
MATERIAL AND SPATIAL DESCRIPTION

In continuum mechanics the ***material description***, or ***Lagrangian description*** , signifies the study of physical or mechanical phenomena by observing what happens to ***a particle*** P ***of the body***.

Alternatively the ***spatial description***, or ***Eulerian description***, consists of observing the events ***occurring at a fixed point in space***. Thus, when the events at all fixed points in space are recorded, we obtain the spatial description.

For simplicity we consider the same coordinate system with the same origin and base vectors $\boldsymbol{e}_i, \ (i = 1, 2, 3)$ to describe the motion in both descriptions.



It is practical for the problems in solid mechanics to formulate and solve in a material description while those in fluid mechanics are easier in a spatial description.

Material and spatial descriptions for a flow represented by the arrows

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MATERIAL AND SPATIAL DESCRIPTION

reference configuration : By definition, it is a particular configuration $\mathcal{R}_r$ used to identify each particle of $\mathcal{B}$. It is easier to define the reference configuration to the initial one $\mathcal{R}_0$ of $\mathcal{B}$ at t = 0.

Material description:
description in which the components of the initial vector position $\boldsymbol{X}$ are independent spatial variables.
Spatial description:
description in which the components of the vector position at later times $\boldsymbol{x}$ are independent spatial variables.

Convention for simplification

functions written with ***small letters*** refer to functions of ***spatial variables***, for example, $f(\boldsymbol{x}, t)$;
functions written with ***capital letters*** refer to functions of ***material variables***, for example, $F(\boldsymbol{X}, t)$.

$$f(\boldsymbol{x},t) \longrightarrow f\big(\chi(\boldsymbol{X},t),t\big) = F(\boldsymbol{X},t)$$

and

$$F(\boldsymbol{X},t) \longrightarrow F\big(\chi^{-1}(\boldsymbol{x},t),t\big) = f(\boldsymbol{x},t)$$

Spatial description:
$\boldsymbol{x}$, *t* are the independant variables.
Material description:
$\boldsymbol{X}$, *t* are the independent variables.

For the partial derivatives we can use either the material coordinates or the spatial coordinates and relate the derivatives of the function with respect to these variables using the chain rule for derivatives of composite functions.

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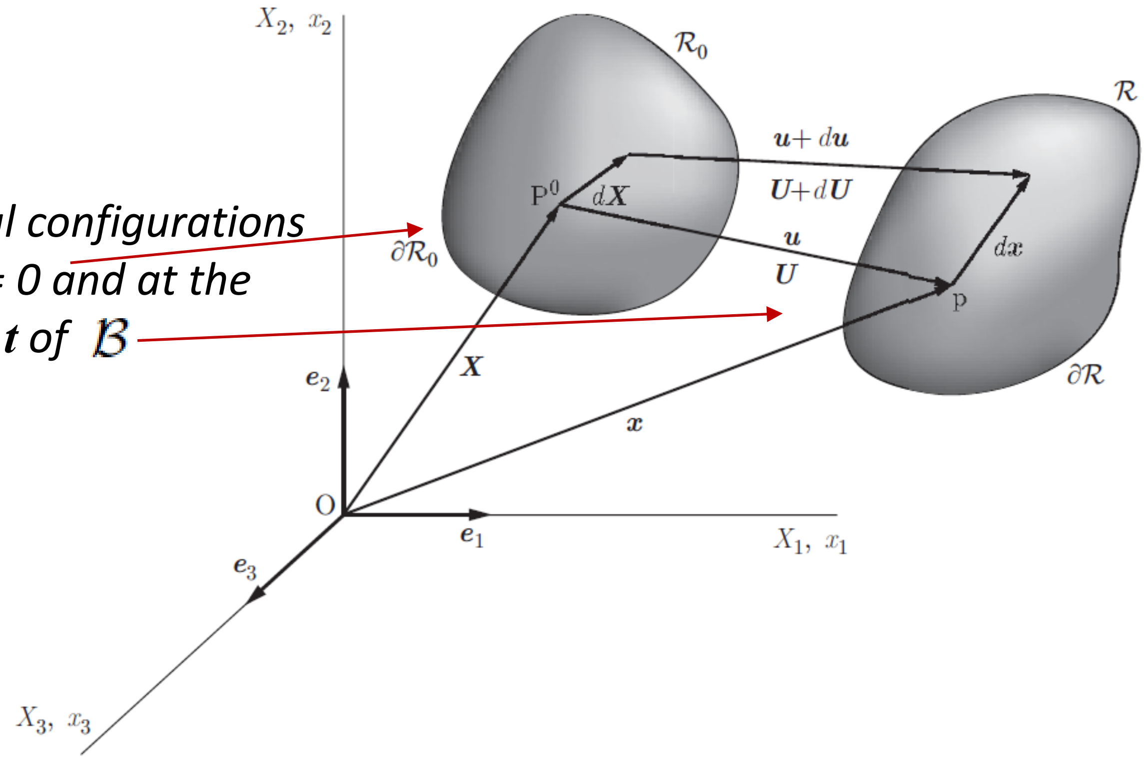
CONFIGURATION AND MOTION

*Initial configurations at **t** = 0 and at the time **t** of* $\mathcal{B}$



A particle is initially ($t = 0$) at $\boldsymbol{X}$ and after a time t is at position $\boldsymbol{x}$.

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X}, t)$$

Displacement in material coordinates

In spatial coordinates the displacement is

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U}\left(\chi^{-1}(\boldsymbol{x}, t), t\right) = \boldsymbol{U}(\boldsymbol{X}, t)$$

The two displacement vectors are equal.

Inverse transformation

$$\boldsymbol{X} = \chi^{-1}(\boldsymbol{x}, t)$$

$$\boldsymbol{x} = \chi^{-1}(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t)$$

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Velocity of a material particle

The material description of velocity of a material particle at time t is the derivative of the motion function with respect to time:

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) \Rightarrow \boldsymbol{V}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial t}$$

The vector $\boldsymbol{V}(\boldsymbol{X}; t)$ expresses the velocity at time t of the particle that initially was at $\boldsymbol{X}$.

Using:

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X}, t)$$

We obtain the velocity at time t in terms of displacement:

$$\boldsymbol{V}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{U}(\boldsymbol{X}, t)}{\partial t}$$

The spatial description of velocity $\boldsymbol{v}(\boldsymbol{x}; t)$ is given by:

$$\boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{V}\left(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), t\right) = \boldsymbol{V}(\boldsymbol{X}, t)$$

Material derivative for a spatial field

Let $\varphi(\boldsymbol{x}, t)$ be a scalar field of $\mathcal{B}$. During a motion material derivative of $\varphi(\boldsymbol{x}, t)$ written as:

$\dot{\varphi}$ or $D\varphi/Dt$

is the rate of change of $\varphi(\boldsymbol{x}, t)$ with time (the derivative with respect to time) for a single particle of $\mathcal{B}$.

$$\begin{aligned}\frac{D\varphi(\boldsymbol{x}, t)}{Dt} &= \frac{\partial \varphi(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \nabla \varphi(\boldsymbol{x}, t) \\ &= \frac{\partial \varphi(\boldsymbol{x}, t)}{\partial t} + v_j \frac{\partial \varphi(\boldsymbol{x}, t)}{\partial x_j} .\end{aligned}$$

The derivative $D\varphi(\boldsymbol{x}, t)/Dt$ is the material derivative. It represents the rate of change of the function $\varphi(\boldsymbol{x}, t)$ following the same particle whose velocity is $\boldsymbol{v}(\mathrm{x}; t)$. Alternatively, this derivative can be considered as giving the change of $\varphi(\boldsymbol{x}, t)$ over time, as seen by an observer moving with the particle that is at $\boldsymbol{x}$.

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Material derivative for a vector field $\boldsymbol{w}$

For a vector field $\boldsymbol{w}$ we have:

$$\frac{D\boldsymbol{w}}{Dt} = \dot{\boldsymbol{w}} = \left.\frac{\partial \boldsymbol{W}(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}=\boldsymbol{\chi}^{-1}(\boldsymbol{x},t)}$$

$$\frac{Dw_i}{Dt} = \dot{w}_i = \left.\frac{\partial W_i(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}=\boldsymbol{\chi}^{-1}(\boldsymbol{x},t)}$$

$$\dot{\boldsymbol{w}} = \frac{\partial \boldsymbol{w}(\boldsymbol{x},t)}{\partial t} + (\nabla \boldsymbol{w}(\boldsymbol{x},t))\left.\frac{\partial \boldsymbol{\chi}(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}=\boldsymbol{\chi}^{-1}(\boldsymbol{x},t)}$$

$$\dot{w}_i = \frac{\partial w_i(\boldsymbol{x},t)}{\partial t} + \frac{\partial w_i(\boldsymbol{x},t)}{\partial x_j} v_j \,.$$

Material derivative for acceleration

The acceleration $\boldsymbol{A}$ is defined as the material derivative of the velocity $\boldsymbol{V}$ with respect to time *t* at time *t.*

In material description:

$$\boldsymbol{A}(\boldsymbol{X},t) = \frac{\partial \boldsymbol{V}(\boldsymbol{X},t)}{\partial t} = \frac{\partial^2 \boldsymbol{\chi}(\boldsymbol{X},t)}{\partial t^2}$$

$$A_i = \dot{V}_i = \frac{\partial^2 \chi_i(\boldsymbol{X},t)}{\partial t^2} \,,$$

In spatial description:

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + (\boldsymbol{\nabla} \boldsymbol{v}(\boldsymbol{x},t))\boldsymbol{v}(\boldsymbol{x},t)$$

$$a_i = \dot{v}_i = \frac{\partial v_i(\boldsymbol{x},t)}{\partial t} + \frac{\partial v_i(\boldsymbol{x},t)}{\partial x_j} v_j(\boldsymbol{x},t)$$

due to the time dependence of $\boldsymbol{v}$ at a fixed point in space (local).

due to the heterogeneity of the velocity field. (advective).

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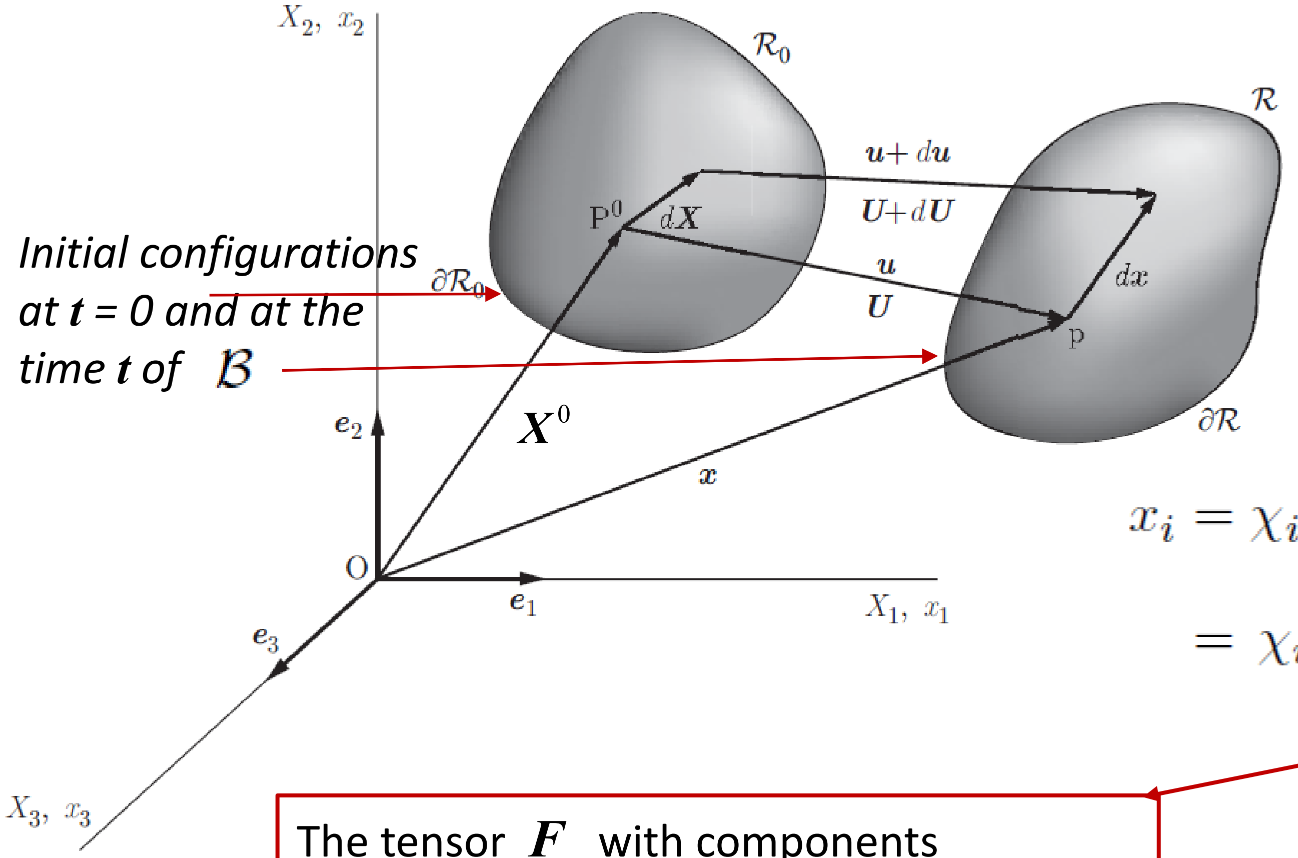
Deformation gradient tensor



Initial configurations at ***t*** *= 0 and at the time* ***t*** *of* $\mathcal{B}$

We consider a particle in configuration $\mathcal{R}_0$ with position $\boldsymbol{X}^0$ and a small neighborhood around it $\mathcal{V}$. .

Its motion is given by, $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$

For a sufficiently small $\mathcal{V}$. the motion for each particle in $\mathcal{V}$ is approximated by a Taylor series around $\boldsymbol{X}^0$ as follows

$$x_i = \chi_i(X_k, t)$$
$$= \chi_i(X_k^0, t) + \left.\frac{\partial \chi_i}{\partial X_j}\right|_{X_k^0} (X_j - X_j^0) + O(\|\boldsymbol{X} - \boldsymbol{X}^0\|^2)$$

The tensor $\boldsymbol{F}$ with components

$$F_{ij} = \frac{\partial \chi_i}{\partial X_j}$$

is called the Deformation gradient tensor .

where

$$O\left(\|\boldsymbol{X} - \boldsymbol{X}^0\|^2\right) \sim C\|\boldsymbol{X} - \boldsymbol{X}^0\|^2 + \cdots$$

and C being a bounded constant.

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<u>Deformation gradient tensor</u>

If $\|\boldsymbol{X} - \boldsymbol{X}^0\| \ll 1$

$$x_i = \chi_i(X_k, t)$$
$$= \chi_i(X_k^0, t) + \left.\frac{\partial \chi_i}{\partial X_j}\right|_{X_k^0} (X_j - X_j^0) + O(\|\boldsymbol{X} - \boldsymbol{X}^0\|^2)$$

⇒ $x_i \cong x_i^0 + F_{ij}(X_j - X_j^0)$ with $x_i^0 = \chi_i(X_k^0, t)$

or $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$

and for simplicity $F_{ij} = \dfrac{\partial x_i}{\partial X_j}$

To assure the continuity of the material and the existence of continuous derivative the Jacobian J of $\boldsymbol{F}$ defined as:

$$J = \det\left(\frac{\partial \chi_i}{\partial X_j}\right) = \det \boldsymbol{F}$$

should satisfy the condition:

$$0 < J < \infty$$

which ensures the existence of the inverse $\boldsymbol{F}^{-1}$ of $\boldsymbol{F}$ with det $\boldsymbol{F} = 1/J$.

we use the motion

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X}, t)$$
$$\boldsymbol{x} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t)$$

⇐ $F_{ij} = \dfrac{\partial x_i}{\partial X_j}$ to obtain

$$F_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial X_j}$$

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j}$$

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Using the polar decomposition theorem, we can express
The deformation gradient tensor $\boldsymbol{F}$ as follows:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$$

right polar decomposition ($\boldsymbol{R}\boldsymbol{U}$)

left polar decomposition ($\boldsymbol{V}\boldsymbol{R}$)

The three tensors are unique:
$\boldsymbol{R}$ expresses a rotation; $\boldsymbol{U}$ and $\boldsymbol{V}$ are called the right and left stretch tensors:

when $\boldsymbol{R} = \boldsymbol{I}$ ⇒ $\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V}$

we have pure deformation.

From:

$$d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X} \Rightarrow d\boldsymbol{x} = \boldsymbol{R}\boldsymbol{U}\, d\boldsymbol{X}$$

⇓

the configuration change in the neighborhood of the material particle is obtained by the transformation of vector $d\boldsymbol{X}$ to a vector $\boldsymbol{U}d\boldsymbol{X}$ by a pure deformation $\boldsymbol{U}$ followed by a local rotation $\boldsymbol{R}$.

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Deformation tensors

Using $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$ in index form we have:

$$dx_i = F_{ij}\, dX_j$$

We can define the square ds o the vector $d\boldsymbol{x}$ as

$$ds^2 = \|d\boldsymbol{x}\|^2 = dx_m\, dx_m = F_{mi} F_{mj}\, dX_i\, dX_j$$

From this expression we can define the following tensor:

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \left(\boldsymbol{F}^T \boldsymbol{F}\right)^T \qquad C_{ij} = F_{mi} F_{mj}$$

Which is defined as the symmetric
right Cauchy-Green deformation tensor.

It is a **metric tensor** in that it can me used to calculate the length of $d\boldsymbol{x}$ as a function of the components $d\boldsymbol{X}$.

We can also calculate $d\boldsymbol{X}$ in terms of $d\boldsymbol{x}$ as follows:

with:

$$dX = F^{-1} dx \;\; ; \;\; dX_m = F^{-1}_{mi} dx_i$$

⇩

$$dS^2 = \|d\boldsymbol{X}\|^2 = dX_m\, dX_m = F^{-1}_{mi} F^{-1}_{mj}\, dx_i\, dx_j$$

and $\quad F^{-1}_{mi} F^{-1}_{mj} = \left(F^T\right)^{-1}_{im} F^{-1}_{mj}$

we define the tensor:

$$\boldsymbol{c}^{-1} = \boldsymbol{F}^{-T} \boldsymbol{F}^{-1} = \left(\boldsymbol{F}^{-T} \boldsymbol{F}^{-1}\right)^T$$

or $\quad c^{-1}_{ij} = F^{-1}_{mi} F^{-1}_{mj}$;

is the inverse of the **symmetric left Cauchy-Green deformation tensor** $\boldsymbol{c}$.

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Deformation tensors

The two tensors

1: symmetric right Cauchy-Green deformation tensor

2: symmetric left Cauchy-Green tensor

are used to express the difference between the squares of the norms $\|d\boldsymbol{x}\|^2$ and $\|d\boldsymbol{X}\|^2$ as follows:

$$\|d\boldsymbol{x}\|^2 - \|d\boldsymbol{X}\|^2 =$$

$$= C_{ij}\, dX_i\, dX_j - dX_m\, dX_m = 2E_{ij}\, dX_i\, dX_j$$

$$\|d\boldsymbol{x}\|^2 - \|d\boldsymbol{X}\|^2 =$$

$$= dx_m\, dx_m - c_{ij}^{-1}\, dx_i\, dx_j = 2e_{ij}\, dx_i\, dx_j$$

The two new tensors are:

1: the Green-Lagrange strain tensor $E_{ij} = \frac{1}{2}\left(C_{ij} - \delta_{ij}\right)$

2: the Euler-Almansi strain tensor $e_{ij} = \frac{1}{2}\left(\delta_{ij} - c_{ij}^{-1}\right)$

These tensors can also be written in terms of displacements $\boldsymbol{U}$ or $\boldsymbol{u}$:

$$C_{ij} = F_{mi}F_{mj} = \left(\delta_{mi} + \frac{\partial U_m}{\partial X_i}\right)\left(\delta_{mj} + \frac{\partial U_m}{\partial X_j}\right)$$

$$= \delta_{ij} + \frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_m}{\partial X_i}\frac{\partial U_m}{\partial X_j}$$

$$c_{ij}^{-1} = F_{mi}^{-1}F_{mj}^{-1} = \left(\delta_{mi} - \frac{\partial u_m}{\partial x_i}\right)\left(\delta_{mj} - \frac{\partial u_m}{\partial x_j}\right)$$

$$= \delta_{ij} - \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_m}{\partial X_i}\frac{\partial U_m}{\partial X_j}\right)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right).$$

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Deformation tensors

The deformation tensors can also be expressed in terms of tensors U and V by applying the polar decomposition:

$$F = RU = VR$$

1: the right Cauchy-Green deformation tensor:

$$C = F^T F = U R^T R U = U^2$$

2: the left Cauchy-Green deformation tensor:

$$c = F F^T = V R R^T V^T = V^2$$

$$c^{-1} = F^{-T} F^{-1} = V^{-2} \; ;$$

3: the Green-Lagrange strain tensor:

$$E = \frac{1}{2}(C - I) = \frac{1}{2}\left(U^2 - I\right)$$

4: the Euler-Almansi strain tensor:

$$e = \frac{1}{2}(I - c^{-1}) = \frac{1}{2}\left(I - V^{-2}\right)$$

Note that the rotation R does not affect the deformation and strain tensors.
(very important in continuum mechanics)

Also when $F = Q$ we have a rigid body motion easily shown below:

From $F = RU$ we have:

$$Q = RU \;\Rightarrow\; R^{-1}Q = R^{-1}RU \;\Rightarrow\; R^{-1}Q = IU$$

without loss of generality we set:

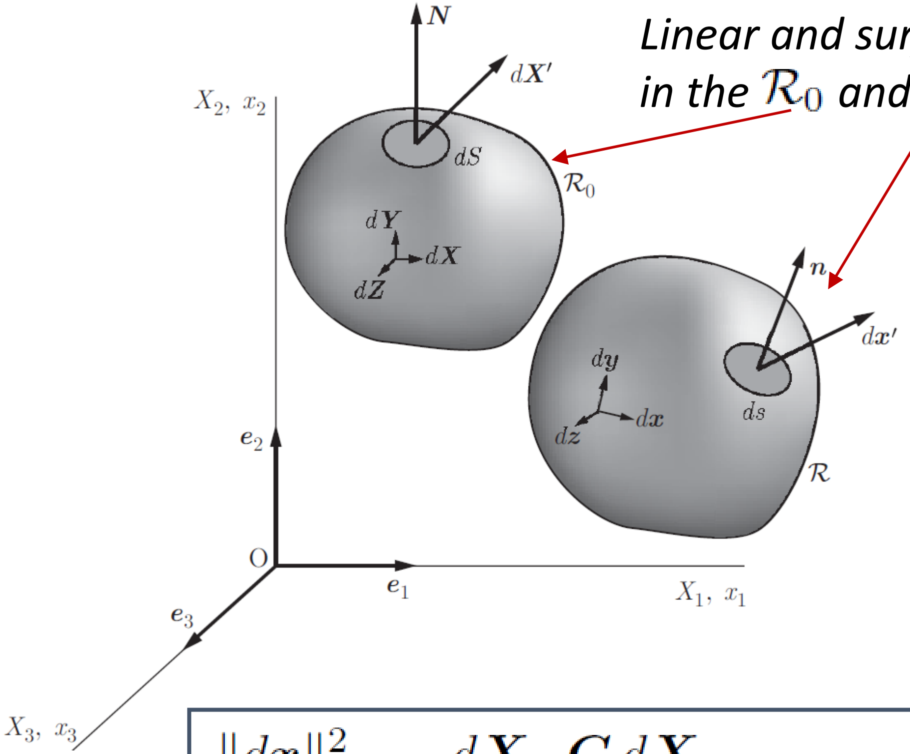
$$Q = R$$

Because they are both orthogonal tensors

$$\Rightarrow\; U = I \;\Rightarrow\; E = 0$$

and similarly $V = I \;\Rightarrow\; e = 0$.

Continuum mechanics review: kinematics



Description of a linear element in two configurations

Using $\boldsymbol{F}$ and the deformation tensors we can express The change in length of a linear element:

A linear element $d\boldsymbol{X}$ in the reference configuration has a norm:

$$\|d\boldsymbol{X}\| = (d\boldsymbol{X} \cdot d\boldsymbol{X})^{1/2}$$

After the motion $\boldsymbol{x} = \chi(\boldsymbol{X}, t)$ it becomes the element $d\boldsymbol{x}$ with norm:

$$\|d\boldsymbol{x}\| = (d\boldsymbol{x} \cdot d\boldsymbol{x})^{1/2}$$

Using $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$ and $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ we obtain:

$$\frac{\|d\boldsymbol{x}\|^2}{\|d\boldsymbol{X}\|^2} = \frac{\boldsymbol{F}\, d\boldsymbol{X} \cdot \boldsymbol{F}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|^2} = \frac{d\boldsymbol{X} \cdot \boldsymbol{F}^T \boldsymbol{F}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|^2} = \frac{d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|^2}.$$

$$d\boldsymbol{X} = \boldsymbol{N}\|d\boldsymbol{X}\|$$

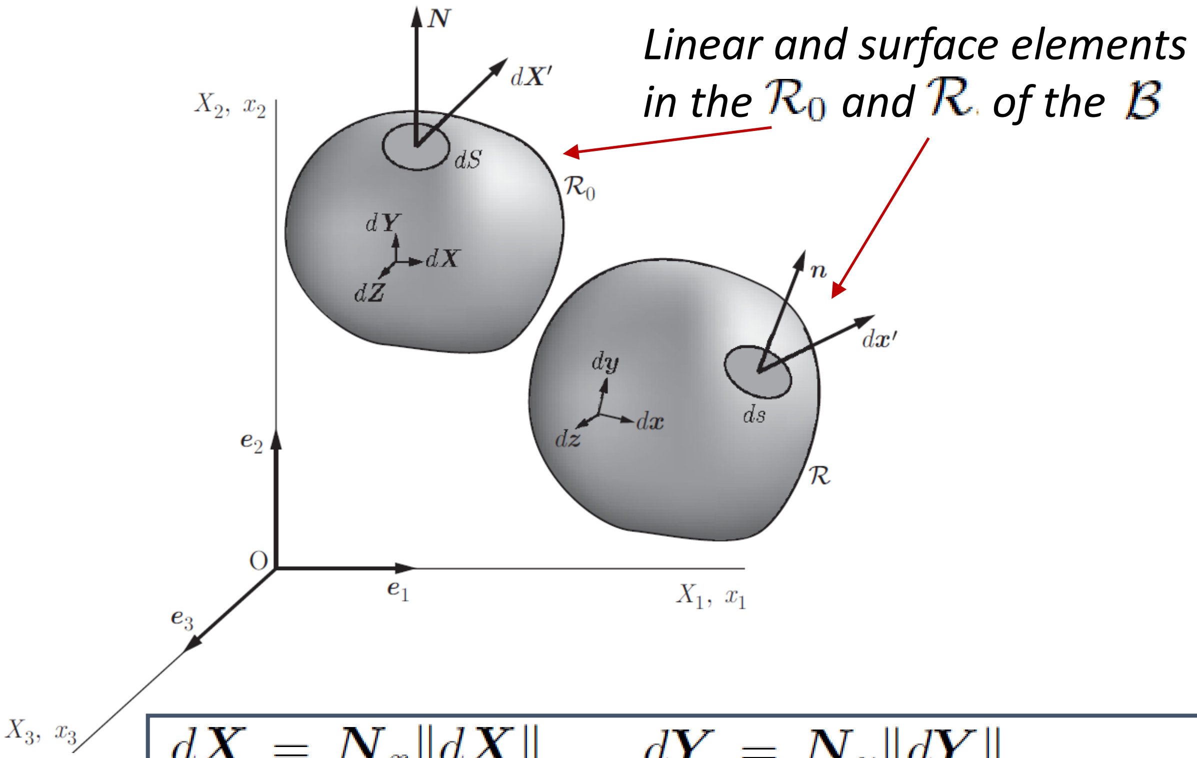
$$\frac{\|d\boldsymbol{x}\|^2}{\|d\boldsymbol{X}\|^2} = \frac{d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{X}}{\|d\boldsymbol{X}\|\,\|d\boldsymbol{X}\|} = \boldsymbol{N} \cdot \boldsymbol{C}\boldsymbol{N} = \lambda_N^2$$

λ_N is the stretch ratio at $\boldsymbol{X}$ in the direction $\boldsymbol{N}$

$$C = U^2$$

$$\frac{\|d\boldsymbol{x}\|}{\|d\boldsymbol{X}\|} = (\boldsymbol{N} \cdot \boldsymbol{U}^2\boldsymbol{N})^{1/2} = (\boldsymbol{U}\boldsymbol{N} \cdot \boldsymbol{U}\boldsymbol{N})^{1/2} = \|\boldsymbol{U}\boldsymbol{N}\| = \lambda_N.$$

Continuum mechanics review: kinematics



$$d\boldsymbol{X} = \boldsymbol{N}_x \|d\boldsymbol{X}\| \qquad d\boldsymbol{Y} = \boldsymbol{N}_y \|d\boldsymbol{Y}\|$$

$d\boldsymbol{X}$ and $d\boldsymbol{Y}$ are unit vectors along $\boldsymbol{X},\ \boldsymbol{Y}$

$$\|\boldsymbol{F}\, d\boldsymbol{X}\| = (\boldsymbol{F}\, d\boldsymbol{X} \cdot \boldsymbol{F}\, d\boldsymbol{X})^{1/2} = (d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{X})^{1/2}$$

$$\cos\theta = \frac{\boldsymbol{N}_x \cdot \boldsymbol{C}\boldsymbol{N}_y}{(\boldsymbol{N}_x \cdot \boldsymbol{C}\boldsymbol{N}_x)^{1/2}(\boldsymbol{N}_y \cdot \boldsymbol{C}\boldsymbol{N}_y)^{1/2}}$$

The difference $\Theta - \theta$ is attributed to shear.

Description of the angle between two linear elements in two configurations

For two linear elements $d\boldsymbol{X}$ and $d\boldsymbol{Y}$ in the reference configuration that intersect with angle Θ we have:

$$\cos\Theta = \frac{d\boldsymbol{X} \cdot d\boldsymbol{Y}}{\|d\boldsymbol{X}\|\, \|d\boldsymbol{Y}\|}$$

After the motion these two elements become $d\boldsymbol{x}$ and $d\boldsymbol{y}$ that intersect with angle θ :

$$\cos\theta = \frac{d\boldsymbol{x} \cdot d\boldsymbol{y}}{\|d\boldsymbol{x}\|\, \|d\boldsymbol{y}\|}$$

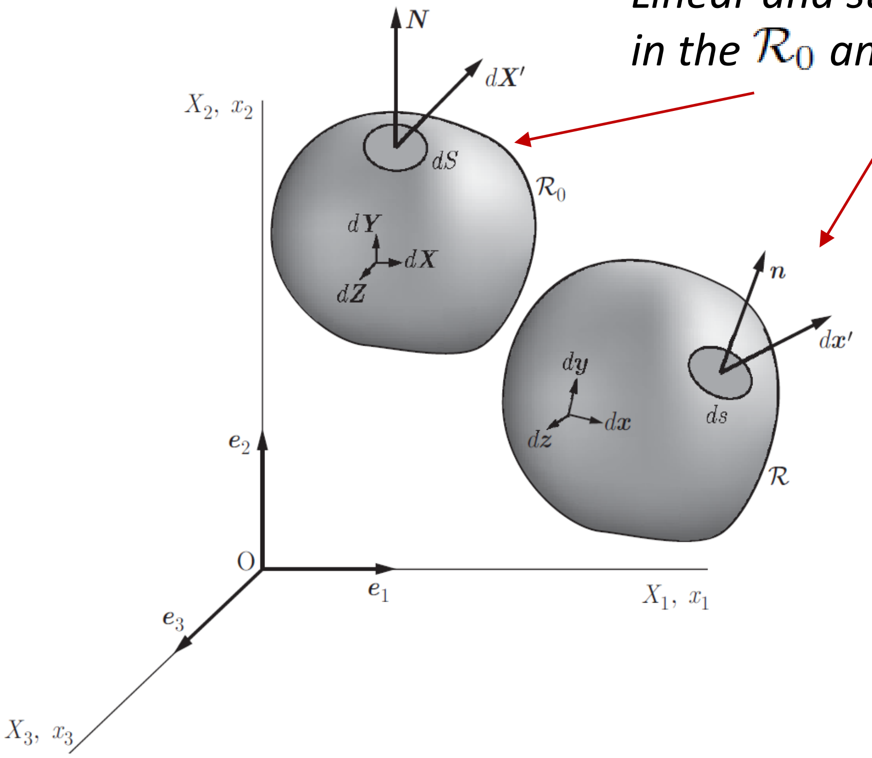
Using $d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}$; $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$; $C = U^2$

$$\cos\theta = \frac{\boldsymbol{F}\, d\boldsymbol{X} \cdot \boldsymbol{F}\, d\boldsymbol{Y}}{\|\boldsymbol{F}\, d\boldsymbol{X}\|\, \|\boldsymbol{F}\, d\boldsymbol{Y}\|} = \frac{d\boldsymbol{X} \cdot \boldsymbol{F}^T \boldsymbol{F}\, d\boldsymbol{Y}}{\|\boldsymbol{F} d\boldsymbol{X}\|\, \|\boldsymbol{F}\, d\boldsymbol{Y}\|}$$

$$= \frac{d\boldsymbol{X} \cdot \boldsymbol{C}\, d\boldsymbol{Y}}{\|\boldsymbol{F}\, d\boldsymbol{X}\|\, \|\boldsymbol{F}\, d\boldsymbol{Y}\|}\,.$$

Continuum mechanics review: kinematics

Linear and surface elements in the $\mathcal{R}_0$ and $\mathcal{R}$ of the $\mathcal{B}$



$$dv = \det \boldsymbol{F}\, dV = J\, dV$$

<u>Description of volume element between two configurations</u>

Consider three non-coplanar linear elements: $d\boldsymbol{X}, d\boldsymbol{Y}$, and $d\boldsymbol{Z}$. We have:

$$dV = d\boldsymbol{X} \cdot (d\boldsymbol{Y} \times d\boldsymbol{Z}) > 0$$

In the deformed configuration, the three linear elements become $d\boldsymbol{x},\ d\boldsymbol{y}$ and $d\boldsymbol{z}$ and the volume is:

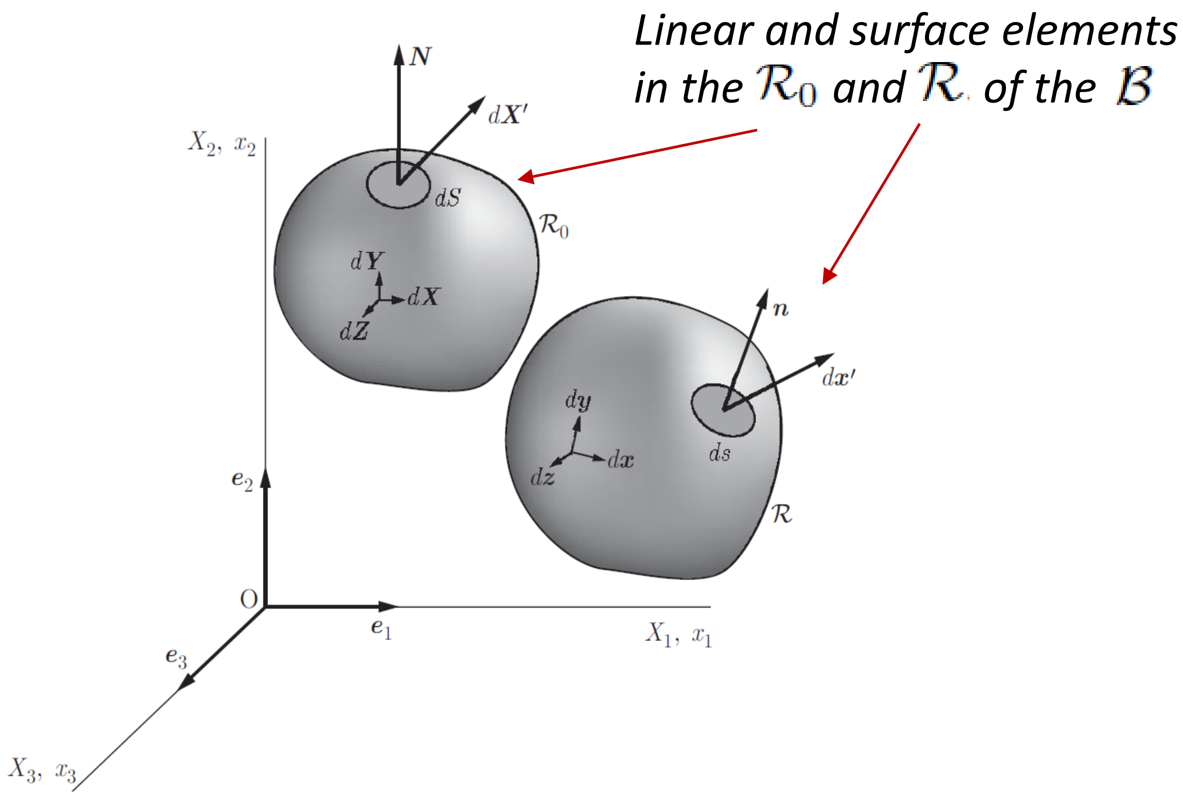
$$dv = d\boldsymbol{x} \cdot (d\boldsymbol{y} \times d\boldsymbol{z})$$

We know that $dx_i = F_{ij}\, dX_j$.

$$dv = \det \begin{pmatrix} dx_1 & dy_1 & dz_1 \\ dx_2 & dy_2 & dz_2 \\ dx_3 & dy_3 & dz_3 \end{pmatrix} =$$

$$= \det \begin{pmatrix} F_{1j}\, dX_j & F_{1j}\, dY_j & F_{1j}\, dZ_j \\ F_{2j}\, dX_j & F_{2j}\, dY_j & F_{2j}\, dZj \\ F_{3j}\, dX_j & F_{3j}\, dY_j & F_{3j}\, dZ_j \end{pmatrix}$$

Continuum mechanics review: kinematics



$$\boldsymbol{n}\,ds = J\boldsymbol{F}^{-T}\boldsymbol{N}\,dS$$

or

$$d\boldsymbol{s} = J\boldsymbol{F}^{-T}\boldsymbol{N}\,dS$$

known as Nanson's formula

Description of surface element between two configurations (Nanson's formula)

To express the change in a surface element we start with the volume element in the reference and deformed configurations:

$$dV = d\boldsymbol{X}' \cdot \boldsymbol{N}\,dS \qquad dv = d\boldsymbol{x}' \cdot \boldsymbol{n}\,ds$$

using $d\boldsymbol{x} = \boldsymbol{F}\,d\boldsymbol{X}$

Relation $dv = \det \boldsymbol{F}\,dV = J\,dV$ becomes

$$dv = \boldsymbol{F}\,d\boldsymbol{X}' \cdot \boldsymbol{n}\,ds = J\,d\boldsymbol{X}' \cdot \boldsymbol{N}\,dS$$

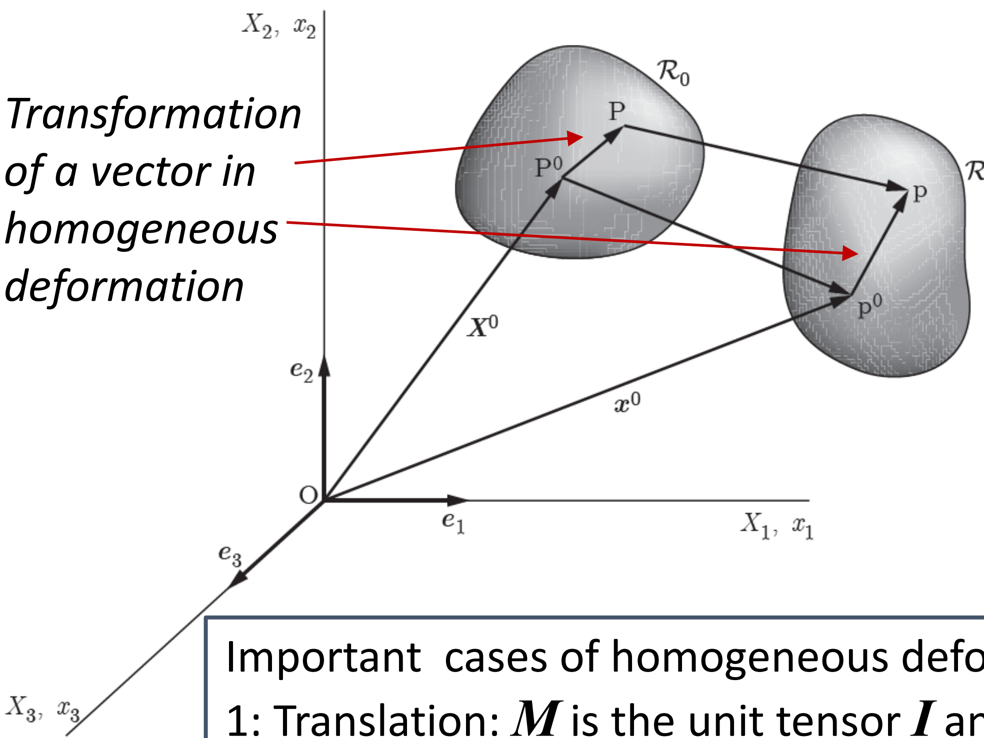
or

$$\left(\boldsymbol{F}^T\boldsymbol{n}\,ds - J\,\boldsymbol{N}\,dS\right) \cdot d\boldsymbol{X}' = 0$$

which is valid for any arbitrary vector $d\boldsymbol{X}'$

Continuum mechanics review: kinematics

Transformation of a vector in homogeneous deformation



Important cases of homogeneous deformation

1: Translation: $\boldsymbol{M}$ is the unit tensor $\boldsymbol{I}$ and if $\boldsymbol{X}^0$ =0

$$\boldsymbol{x} = \boldsymbol{x}^0(t) + \boldsymbol{X}$$

2: Rotation about the origin: $\boldsymbol{X}^0 = \boldsymbol{x}^0 = \boldsymbol{0}$

and $\boldsymbol{M}$ is the rotation tensor $\boldsymbol{R}$ ($\boldsymbol{R}^T\boldsymbol{R}=\boldsymbol{R}\boldsymbol{R}^T=\boldsymbol{I}$)

$$\boldsymbol{x} = \boldsymbol{R}\boldsymbol{X} \qquad \boldsymbol{X} = \boldsymbol{R}^T\boldsymbol{x}$$

rigid body motion is decomposed into rotation and translation

Homogeneous deformation

The deformation (or transformation) $\boldsymbol{x}$ of a body $\mathcal{B}$ is defined as **homogeneous** if the corresponding deformation gradient $\boldsymbol{F}$ is independent of the particle's position $\boldsymbol{X}$.

A homogeneous deformation transforms a straight line $\mathrm{P}^0\mathrm{P}$ of $\mathcal{R}_0$ to a straight line $\mathrm{p}^0\mathrm{p}$ of $\mathcal{R}$.

Such a deformation $\boldsymbol{x}$ is an **affine transformation** and has the following general form (with $x_i^0 = \chi_i(X_j^0, t)$) :

$$x_i = x_i^0(t) + M_{ij}(t)\left(X_j - X_j^0\right)$$

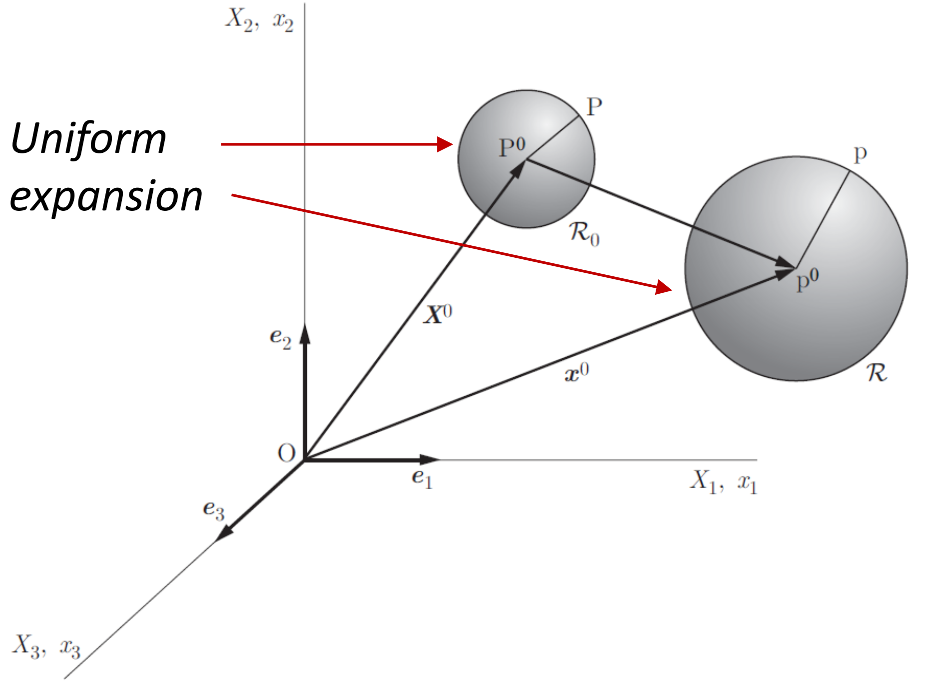
In vector form it is:

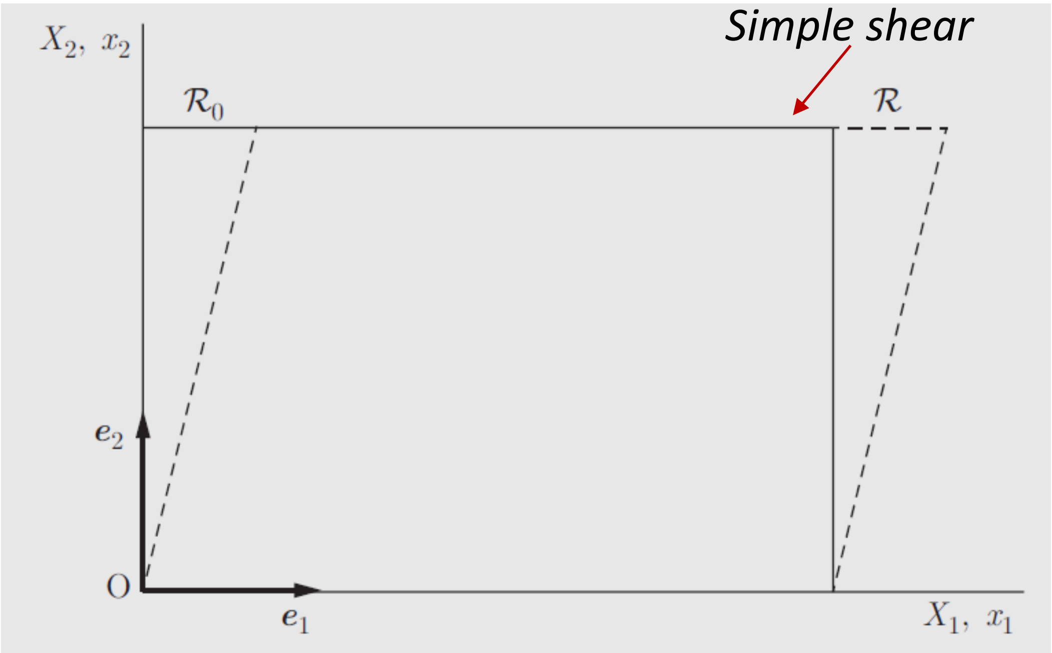
$$\boldsymbol{x} = \boldsymbol{x}^0(t) + \boldsymbol{M}(t)\left(\boldsymbol{X} - \boldsymbol{X}^0\right)$$

with its inverse given by ($0 < \det \boldsymbol{M} < \infty$) :

$$\boldsymbol{X} = \boldsymbol{X}^0 + \boldsymbol{M}^{-1}(t)\left(\boldsymbol{x} - \boldsymbol{x}^0\right)$$

Continuum mechanics review: kinematics





3: Homogeneous deformation: $\boldsymbol{M} = m\boldsymbol{I}$

$$x_i = x_i^0(t) + M_{ij}(t)\big(X_j - X_j^0\big)$$

$$\Rightarrow \quad x_i = x_i^0 + m\big(X_i - X_i^0\big)$$

4: simple shear

in Cartesian coordinates $[M] = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

in vector form $\boldsymbol{x} = \boldsymbol{M}\boldsymbol{X} = (\boldsymbol{I} + k\,\boldsymbol{e}_1 \otimes \boldsymbol{e}_2)\boldsymbol{X}$

with a fixed origin $\boldsymbol{X}^0 = \boldsymbol{x}^0 = \boldsymbol{0}$

$$\Rightarrow \quad x_1 = X_1 + kX_2 \qquad x_2 = X_2 \qquad x_3 = X_3 \,.$$

Continuum mechanics review: kinematics

Small Displacements

Consider a displacement field dependent on a small real number $\varepsilon \ (\varepsilon << 1)$ such that:

Lagrangian — Eulerian

$\boldsymbol{U}(\boldsymbol{X}) = \varepsilon \boldsymbol{W}(\boldsymbol{X})$ where $\boldsymbol{W}(\boldsymbol{X})$ and $\boldsymbol{w}(\boldsymbol{x})$ are known.

From the Green-Lagrange and Euler-Almansi strain tesnors:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_m}{\partial X_i}\frac{\partial U_m}{\partial X_j}\right)$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right).$$

$$E_{ij} = \varepsilon\frac{1}{2}\left(\frac{\partial W_i}{\partial X_j} + \frac{\partial W_j}{\partial X_i}\right) + \varepsilon^2\frac{1}{2}\frac{\partial W_m}{\partial X_i}\frac{\partial W_m}{\partial X_j}$$

$$e_{ij} = \varepsilon\frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) - \varepsilon^2\frac{1}{2}\frac{\partial w_m}{\partial x_i}\frac{\partial w_m}{\partial x_j}.$$

When $\varepsilon \rightarrow 0$ approaches zero we obtain

$$E_{ij} \simeq \varepsilon\frac{1}{2}\left(\frac{\partial W_i}{\partial X_j} + \frac{\partial W_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$$

$$e_{ij} \simeq \varepsilon\frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Continuum mechanics review: kinematics

Kinematic linearization

Using:

$$x_i = X_i + U_i = X_i + \varepsilon W_i$$

$$W_i(X_k) = w_i(x_k)$$

we can write:

$$\frac{\partial U_i}{\partial X_j} = \varepsilon \frac{\partial W_i}{\partial X_j} = \varepsilon \frac{\partial w_i}{\partial x_k} \frac{\partial x_k}{\partial X_j}$$
$$= \varepsilon \frac{\partial w_i}{\partial x_k} \left(\delta_{kj} + \varepsilon \frac{\partial W_k}{\partial X_j} \right)$$
$$= \frac{\partial u_i}{\partial x_j} + \varepsilon^2 \frac{\partial w_i}{\partial x_k} \frac{\partial W_k}{\partial X_j}$$

$$\Rightarrow \quad \frac{\partial U_i}{\partial X_j} = \frac{\partial u_i}{\partial x_j} + O(\varepsilon^2) \simeq \frac{\partial u_i}{\partial x_j}$$

When the displacements are small and $O(\varepsilon^2) \to 0$ approaches zero, the difference between the two strain tensors is negligible.

Based on results in engineering applicatios we have:

$$\left\| \frac{\partial U_i}{\partial X_j} \right\| = O(\varepsilon) \ll 1$$

$$\Downarrow$$

$$F_{ij} = \delta_{ij} + \frac{\partial U_i}{\partial X_j} \Rightarrow F_{ij} = \delta_{ij} + O(\varepsilon)$$

$$F_{ij}^{-1} = \delta_{ij} - \frac{\partial u_i}{\partial x_j} \Rightarrow F_{ij}^{-1} = \delta_{ij} - O(\varepsilon)$$

$$J = \det \boldsymbol{F} \Rightarrow J \approx 1 + O(\varepsilon)$$

Continuum mechanics review: kinematics

Infinitesimal strain tensor

The resulting kinematic linearization:

$$\frac{\partial U_i}{\partial X_j} = \frac{\partial u_i}{\partial x_j} + O(\varepsilon^2) \simeq \frac{\partial u_i}{\partial x_j}$$

shows that if terms of order ε^2 are negilible i.e., $O(\varepsilon^2) \to 0$ there is no difference between Green-Lagrange and Euler-Almansi strain tensors. Therefore, we introduce the infinitesimal strain tensor:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla U} + (\boldsymbol{\nabla U})^T\right) = \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right)$$

Note: $\boldsymbol{\varepsilon}$ is a 2nd order tensor because the displacement gradient is a tensor shown next.

Note that : $(\nabla \boldsymbol{u})_{kl} = \dfrac{\partial u_k}{\partial x_l}$

If it is a 2nd order tensor, it should be transformed as follows: $\dfrac{\partial u_i'}{\partial x_j'} = c_{ik} c_{jl} \dfrac{\partial u_k}{\partial x_l}$

We know that,

$$u_i' = c_{ik} u_k \Rightarrow \frac{\partial u_i'}{\partial x_j'} = c_{ik} \frac{\partial u_k}{\partial x_j'} \frac{\partial x_l}{\partial x_l} = c_{ik} \frac{\partial x_l}{\partial x_j'} \frac{\partial u_k}{\partial x_l}$$

$$x_l = c_{jl} x_j' \Rightarrow \frac{\partial x_l}{\partial x_j'} = c_{jl}$$

$$\Rightarrow \frac{\partial u_i'}{\partial x_j'} = c_{ik} c_{jl} \frac{\partial u_k}{\partial x_l}$$

Continuum mechanics review: kinematics

Infinitesimal strain tensor

The resulting kinematic linearization:

$$\frac{\partial U_i}{\partial X_j} = \frac{\partial u_i}{\partial x_j} + O(\varepsilon^2) \simeq \frac{\partial u_i}{\partial x_j}$$

shows that if terms of order ε^2 are negligible

i.e., $O(\varepsilon^2) \to 0$ there is no difference between Green-Lagrange and Euler-Almansi strain tensors. Therefore, we introduce the infinitesimal strain tensor:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\nabla U} + (\boldsymbol{\nabla U})^T\right) = \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right)$$

Note: $\boldsymbol{\varepsilon}$ is a 2nd tensor because the displacement gradient is a tensor.

Since it is tensor the transformation law for its components is given by:

$$L'_{ij} = c_{ik} c_{jl} L_{kl}$$

The eigenvalues, which correspond to the principal infinitesimal strains, are from the solutions of equation:

$$\lambda^3 - I_1(\boldsymbol{L})\lambda^2 + I_2(\boldsymbol{L})\lambda - I_3(\boldsymbol{L}) = 0$$

With $\boldsymbol{L} = \boldsymbol{\varepsilon}$

Note that the tensor $\boldsymbol{\varepsilon}$ is linear in $\nabla \boldsymbol{u}$.

Therefore for the strains $\boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}, \ldots$
resulting from the displacements $\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \ldots$
the principle of superposition applies, i.e., the total strain:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(1)} + \boldsymbol{\varepsilon}^{(2)} + \ldots$$

corresponds to

$$\boldsymbol{u} = \boldsymbol{u}^{(1)} + \boldsymbol{u}^{(2)} + \ldots$$

Continuum mechanics review: kinematics

Infinitesimal strain tensor

Given the displacement field, the strains are calculated frrom:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

However, for a given ε_{ij} , a corresponding displacement field does not necessarily exist. To make sure the displacement field exists, the conditions of integrability should be satisfied. Based on differential calculus these conditions are obtained by differentiating the strain-displacement relations, i.e.,

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0$$

These are the so-called compatibility equations and in explicit for the are six of them:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right)$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(-\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2}\right)$$

$$\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_{11}}{\partial {x_2}^2} + \frac{\partial^2 \varepsilon_{22}}{\partial {x_1}^2}\right)$$

$$\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_{22}}{\partial {x_3}^2} + \frac{\partial^2 \varepsilon_{33}}{\partial {x_2}^2}\right)$$

$$\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_{33}}{\partial {x_1}^2} + \frac{\partial^2 \varepsilon_{11}}{\partial {x_3}^2}\right).$$

The are the **necessary and sufficient** conditions for a unique displacement field when the body is **simply connected**.
for multiply connected elastic solid, they are not sufficient and additional conditions are needed.

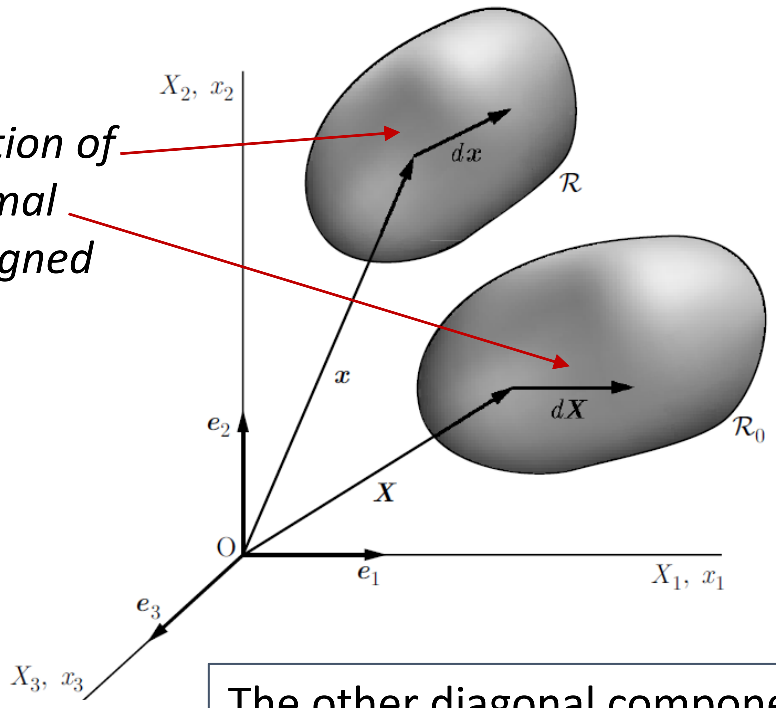
Continuum mechanics review: kinematics

Infinitesimal strain tensor ε_{ij}

1: **Interpretation of the component** ε_{11},

Consider an infinitesimal vector $d\boldsymbol{X}$ attached to the point $\boldsymbol{X}$ with components (dX_1; 0; 0).

Deformation of infinitesimal vector aligned with $\boldsymbol{e}_1$



The length of the corresponding vector $d\boldsymbol{x}$ in $\mathcal{R}$ is given by :

$$\|d\boldsymbol{x}\|^2 - \|d\boldsymbol{X}\|^2 = 2E_{ij}\, dX_i\, dX_j$$

$$\|d\boldsymbol{x}\|^2 = \|d\boldsymbol{X}\|^2 + 2E_{ij}\, dX_i\, dX_j = (1 + 2E_{11})\, dX_1^2$$

(assuming small gradients)

$$\|d\boldsymbol{x}\|^2 \cong (1 + 2\varepsilon_{11})\|d\boldsymbol{X}\|^2$$

$$\|d\boldsymbol{x}\| \cong (1 + 2\varepsilon_{11})^{1/2}\, dX_1 = (1 + \varepsilon_{11})\|d\boldsymbol{X}\|$$

$$\varepsilon_{11} \cong \frac{\|d\boldsymbol{x}\| - \|d\boldsymbol{X}\|}{\|d\boldsymbol{X}\|}$$

relative extension of a material line element aligned with direction 1.

The other diagonal components ε_{22} and ε_{33} of $\boldsymbol{\varepsilon}$ have similar interpretations.

Continuum mechanics review: kinematics

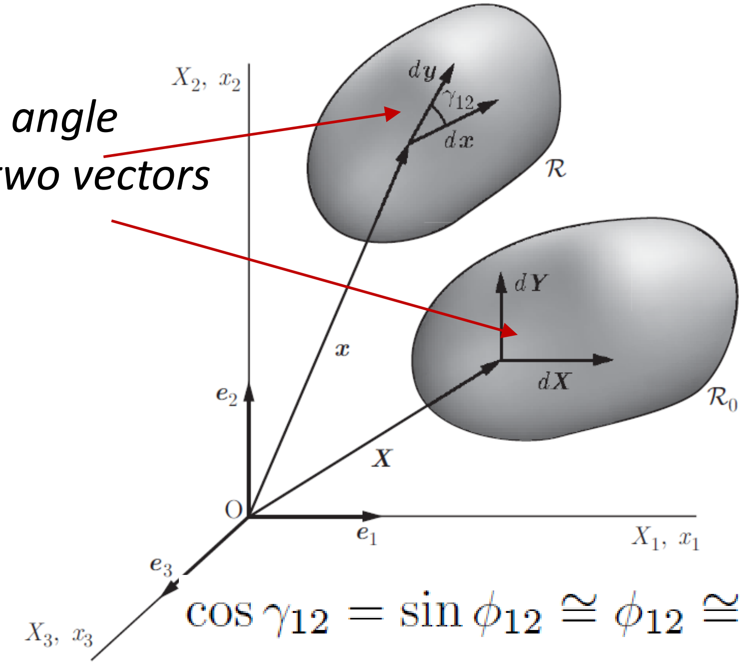
Infinitesimal strain tensor ε_{ij}

2: Interpretation of the component ε_{12}

Consider two orthogonal vectors in $\mathcal{R}_0$:

$$dX = (dX_1, 0, 0) \quad dY = (0, dY_2, 0)$$

Change in angle between two vectors



$$\cos\gamma_{12} = \sin\phi_{12} \cong \phi_{12} \cong 2\varepsilon_{12}$$

Similar interpratation can be given to ε_{23} and ε_{31}.

In $\mathcal{R}$ these vectors are deformed to $d\boldsymbol{x}$, $d\boldsymbol{y}$ with components:

$$dx_i = F_{i1}\, dX_1 \qquad dy_i = F_{i2}\, dY_2$$

The corresponding lengths are:

$$\|d\boldsymbol{x}\| \cong (1 + \varepsilon_{11})\, dX_1$$

$$\|d\boldsymbol{y}\| \cong (1 + \varepsilon_{22})\, dY_2$$

The angle γ_{12} between them is:

$$\cos\gamma_{12} = \frac{d\boldsymbol{x} \cdot d\boldsymbol{y}}{\|d\boldsymbol{x}\|\,\|d\boldsymbol{y}\|}$$

$$\cong \frac{2\varepsilon_{12}}{(1+\varepsilon_{11})(1+\varepsilon_{22})} \cong 2\varepsilon_{12}$$

with $\phi_{12} = \frac{\pi}{2} - \gamma_{12}$ as a slip angle.

Continuum mechanics review: kinematics

Infinitesimal strain tensor ε_{ij}

3: Relative variation of a volume element

We consider three orthogonal vectors:

$$d\boldsymbol{X} = dX\,\boldsymbol{e}_1, d\boldsymbol{Y} = dY\,\boldsymbol{e}_2, d\boldsymbol{Z} = dZ\,\boldsymbol{e}_3$$

In the referenced (undeformed) configuration.

Volume before deformation: $dV = dX\,dY\,dZ$

After deformation:

$$dx = (1+\varepsilon_{11})dX \quad dy = (1+\varepsilon_{22})dY$$

$$dz = (1+\varepsilon_{33})dZ$$

$$\begin{aligned} dv &= dxdydz \\ &= (1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33})dXdYdZ \\ &= (1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33})dV \end{aligned}$$

Neglecting the higher order terms of the deformation:

$$\Rightarrow \frac{dv - dV}{dV} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii}$$

which is the trace of the strain tensor $\boldsymbol{\varepsilon}$.

Recalling the definition of the divergence of a vector field we have:

$$\varepsilon_{ii} = \operatorname{div}\boldsymbol{u} = \operatorname{tr}(\boldsymbol{\nabla}\boldsymbol{u})$$

Continuum mechanics review: kinematics

Infinitesimal strain tensor ε

The matrix form the tensor ε is:

$$[\varepsilon]=\begin{pmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}\end{pmatrix}$$

To obtain the principla strains we solve the characteristic equation:

$$\lambda^3 - I_1(\varepsilon)\lambda^2 + I_2(\varepsilon)\lambda - I_3(\varepsilon) = 0$$

The three strain invariants are,

$$I_1(\varepsilon)=\varepsilon_{ii} \quad \Leftarrow \quad \varepsilon_{ii} = \operatorname{div} \boldsymbol{u} = \operatorname{tr}(\nabla \boldsymbol{u})$$

$$I_2(\varepsilon)=\frac{1}{2}(\varepsilon_{ii}\varepsilon_{jj}-\varepsilon_{ij}\varepsilon_{ji}) \qquad I_3(\varepsilon)=\det \varepsilon$$

i.e., similar to the stress invariants.

The changes of coordinates modify the components of the tensor according to the relation:

$$[\varepsilon']=[C][\varepsilon][C]^T$$

For a plane stress problem, its expilict form is:

$$\begin{pmatrix}\varepsilon'_{11} & \varepsilon'_{12}\\ \varepsilon'_{21} & \varepsilon'_{22}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\varepsilon_{11} & \varepsilon_{12}\\ \varepsilon_{21} & \varepsilon_{22}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

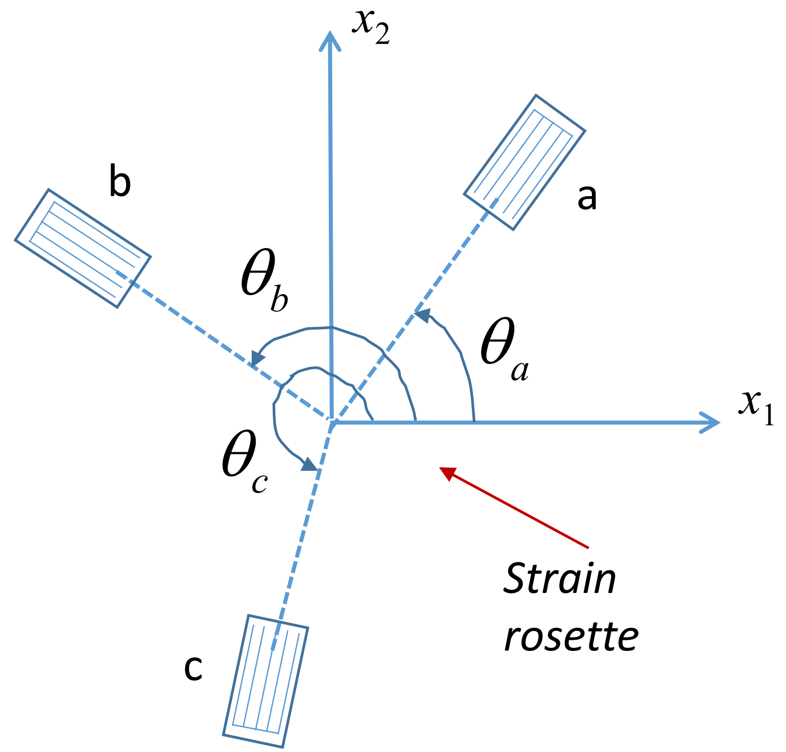
$$\varepsilon'_{11}=\varepsilon_{11}\cos^2\theta+\varepsilon_{22}\sin^2\theta+2\varepsilon_{12}\cos\theta\sin\theta$$
$$\varepsilon'_{22}=\varepsilon_{11}\sin^2\theta+\varepsilon_{22}\cos^2\theta-2\varepsilon_{12}\cos\theta\sin\theta$$
$$\varepsilon'_{12}=(\varepsilon_{22}-\varepsilon_{11})\cos\theta\sin\theta+\varepsilon_{12}(\cos^2\theta-\sin^2\theta)$$

These expressions and the corresponding ones for stresses are similar. It means that we can use Mohr's circle to calculate strains along different directions.

Continuum mechanics review: kinematics

Application: Experimental strain measurements

We can measure strains along the direction of the applied load (in uniaxila traction/compression) with the use of *electrical-resistance strain gauge* (a wire grid or metal foil bonded to the specimen). A combination of strain gauges to measure strains on a surface in different directions exists. We use a cluster of *electrical-resistance strain gauges* arranged in a predetermined pattern *and are called Strain rosettes.* Two commonly used ones are:



$$\varepsilon_a = \varepsilon_{11}\cos^2\theta_a + \varepsilon_{22}\sin^2\theta_a + 2\varepsilon_{12}\cos\theta_a\sin\theta_a$$

$$\varepsilon_b = \varepsilon_{11}\cos^2\theta_b + \varepsilon_{22}\sin^2\theta_b + 2\varepsilon_{12}\cos\theta_b\sin\theta_b$$

$$\varepsilon_c = \varepsilon_{11}\cos^2\theta_c + \varepsilon_{22}\sin^2\theta_c + 2\varepsilon_{12}\cos\theta_c\sin\theta_c$$

square rosette:

$$\theta_a = 0^o \;;\; \theta_b = 45^o \;;\; \theta_c = 90^o$$

$$\varepsilon_{11} = \varepsilon_a \;;\; \varepsilon_{22} = \varepsilon_c \;;\; 2\varepsilon_{12} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)$$

60º rosette:

$$\theta_a = 0^o \;;\; \theta_b = 60^o \;;\; \theta_c = 120^o$$

$$\varepsilon_{11} = \varepsilon_a \;;\; \varepsilon_{22} = \frac{1}{3}(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a) \;;\; 2\varepsilon_{12} = \frac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c)$$

$(\varepsilon_{12} = \varepsilon_{21})$

$$[\varepsilon] = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$$

Continuum mechanics review: kinematics

Infinitesimal rotation tensor $\boldsymbol{\omega}$

3: Relative variation of a volume element

For a displacement field $\boldsymbol{u}$ we have

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial X_i}\right) dx_j$$

or

$$d\boldsymbol{u} = \boldsymbol{\nabla} \boldsymbol{u}\, d\boldsymbol{x} = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right) d\boldsymbol{x} + \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T\right) d\boldsymbol{x}$$

symmetric → $\boldsymbol{\varepsilon}$

antisymmetric → $\boldsymbol{\omega}$

or

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \qquad \boldsymbol{\omega} = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} - (\boldsymbol{\nabla}\boldsymbol{u})^T\right)$$

with $du_i = \varepsilon_{ij}\, dx_j + \omega_{ij}\, dx_j$

The three independent components of the antisymmetric tensor $\boldsymbol{\omega}$ can be expressed as the curl of the displacement vector

$$\frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{u} = \omega_{32}\boldsymbol{e}_1 + \omega_{13}\boldsymbol{e}_2 + \omega_{21}\boldsymbol{e}_3$$

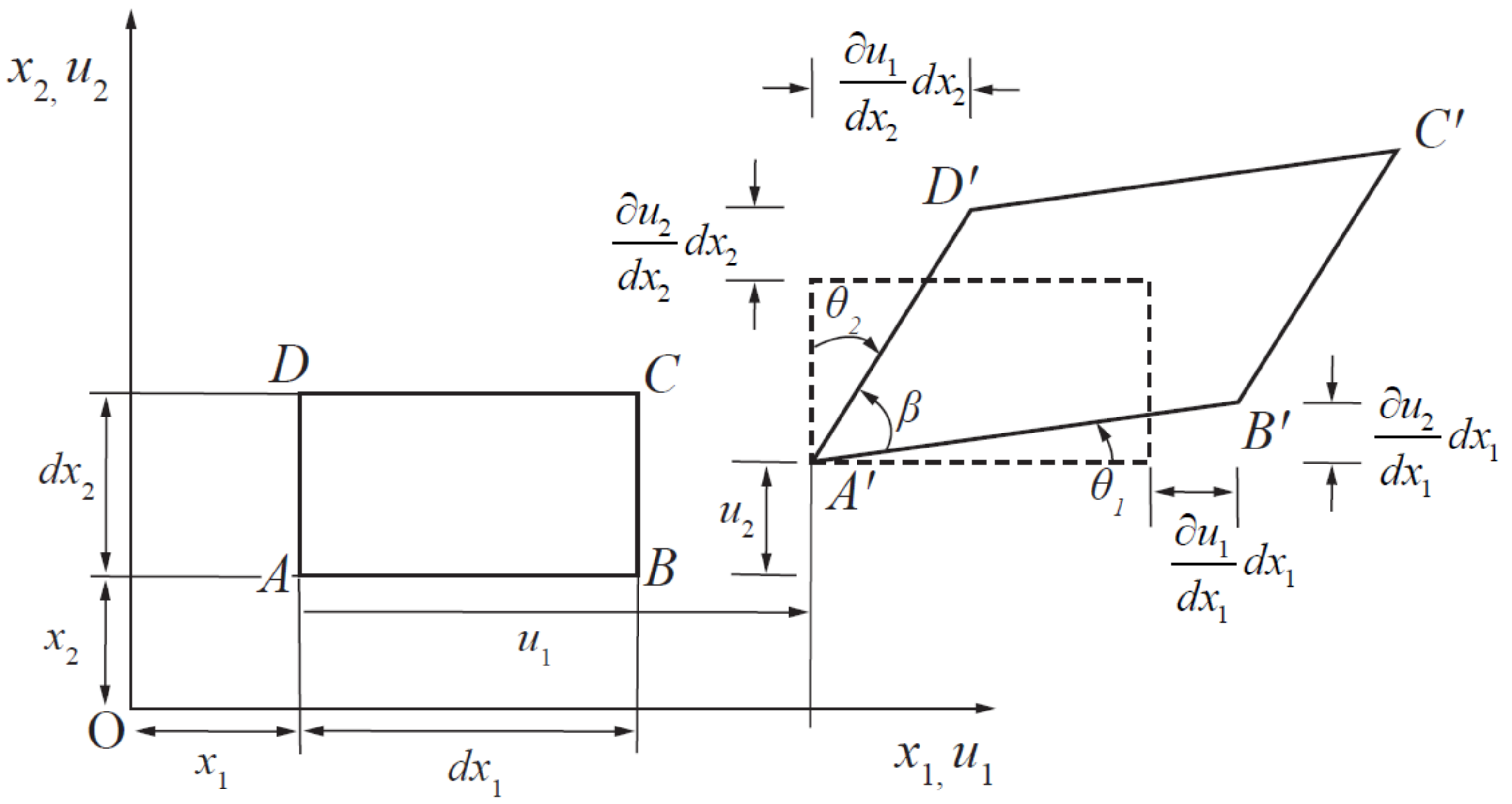
↓

The infinitesimal displacement can be decomposed into a sum of a pure strain tensor and a pure rotation.

An additive decomposition of the displacement gradient is not possible for large strains where $(\boldsymbol{E} \neq \boldsymbol{\varepsilon})$. In this case we should use

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$$

Continuum mechanics review: kinematics 2D strains



$$\theta_1 = \frac{\partial u_2}{\partial x_1}; \quad \theta_2 = -\frac{\partial u_1}{\partial x_2}$$

$$\varphi_{12} = \theta_1 - \theta_2; \quad \beta + \varphi_{12} = \frac{\pi}{2}$$

$$\cos\beta = \sin\varphi_{12} \approx \varphi_{12} =$$

$$\theta_1 - \theta_2 = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = 2\varepsilon_{12}$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}; \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}; \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Continuum mechanics review: kinematics

Objectivity of kinematic parameters

Distinguish between two descriptions

1: change of coordinate system for the same event for a single observer
This development is the basis of tensor analysis and is imposed by the requirement that all laws of continuum physics must be independent of the choice of coordinate system by the observer.

2: change of the observer or reference frame. Here the same event is described in two different reference frames.
A reference frame must have an observer to record the event as well as a coordinate system.
EXAMPLE: Inertial reference frames (a body moves with constant velocity when free of forces). **Newton's 2nd law hold**.
Anther example is an **accelerated reference frames.**
Here **Newton's 2nd law needs to be modified**.
Rotating reference frame. This frame of reference is rotating with respect to an inertial system and requires an additional acceleration). **Newton's 2nd law needs to be modified**.

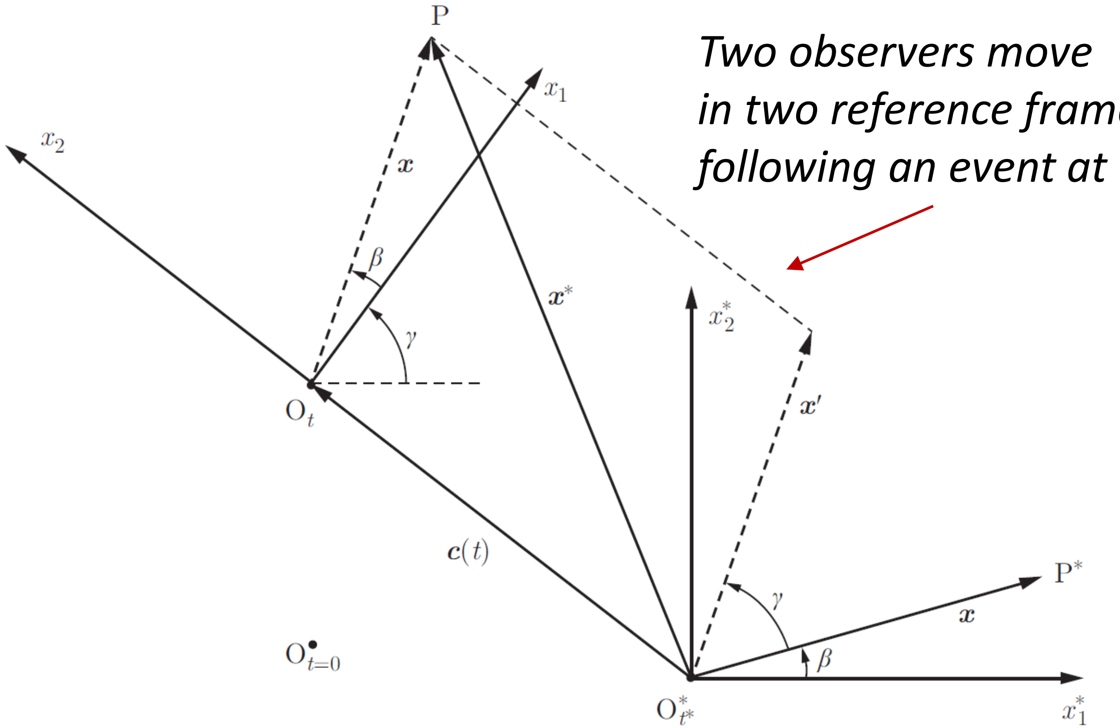
In mechanics we want to distinguish the kinematic parameters, scalars, vectors, or tensors, which depend intrinsically on the observer from those that are essentially independent.

This is particularly important in the description of the constitutive relations with non-linear materials response.

In mechanics of continuous media, an event, that is, a physical process, is defined by its coordinates in space $\boldsymbol{x}$ and the observation time t.

Continuum mechanics review: kinematics

Objectivity of kinematic parameters



Two observers move in two reference frames following an event at P.

The same observation at P (experiment) seen by two observers in the corresponding reference frames at the same time. For the observer at R the vector position is x. For the observer at R^* we must take into account the rotation of R with respect to R^*.

Consider an event viewed by two observers R and R^*, and noted respectively by $(\boldsymbol{x}, t)$ and $(\boldsymbol{x}^*, t^*)$.

The motion between two observers is a function of space and time (effects due to relativity are negligible).

The two observers measure the same distance between two events as well as the same time intervals between events.

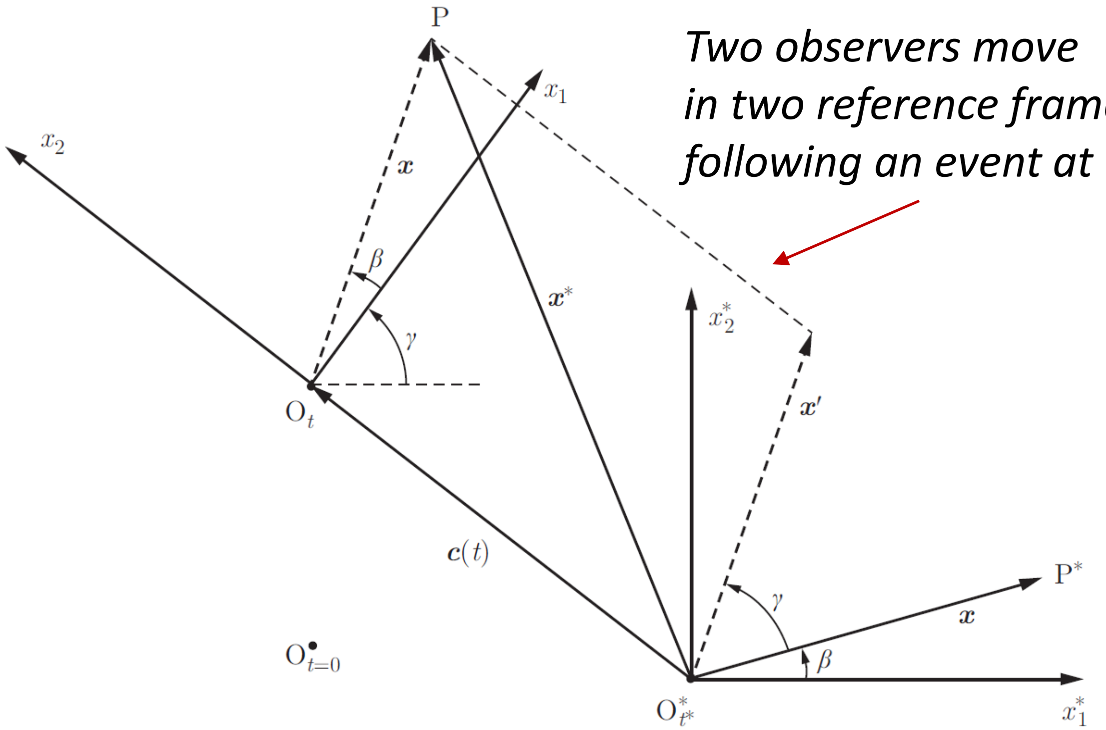
The most general transformation between the two observations of the same event is given by:

$$\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t) \quad \text{where} \quad t^* = t - \alpha$$

Here $\boldsymbol{Q}(t)$ is an orthogonal tensor with time as a parameter, $\boldsymbol{c}(t)$ is a vector and α is a scalar.

Continuum mechanics review: kinematics

Objectivity of kinematic parameters



Two observers move in two reference frames following an event at P.

$$\|\boldsymbol{u}^*\| = \|\boldsymbol{u}\|$$

$$\boldsymbol{u}^* \cdot \boldsymbol{u}^* = (\boldsymbol{Q}\boldsymbol{u}) \cdot (\boldsymbol{Q}\boldsymbol{u}) = \boldsymbol{u} \cdot (\boldsymbol{Q}^T\boldsymbol{Q})\boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{u}$$

The transformation $\boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u}$ is that of a rigid body.

The motion of the body $\mathcal{B}$ described by $\chi(\boldsymbol{X}, t)$ according to the first observer, is described by the second observer as $\chi^*(\boldsymbol{X}, t^*)$.

The two descriptions are related as follows:

$$\chi^*(\boldsymbol{X}, t^*) = \boldsymbol{Q}(t)\chi(\boldsymbol{X}, t) + \boldsymbol{c}(t)$$

To examine the ramifications of this relation we consider two events reordered by:

$$\mathrm{R} : (\boldsymbol{x}_1, t), (\boldsymbol{x}_2, t) ; \mathrm{R}^* : (\boldsymbol{x}_1^*, t) \ (\boldsymbol{x}_2^*, t)$$

The relative positions of these events are:

$$\mathrm{R} : \boldsymbol{u} = \boldsymbol{x}_2 - \boldsymbol{x}_1 ; \mathrm{R}^* : \boldsymbol{u}^* = \boldsymbol{x}_2^* - \boldsymbol{x}_1^*$$

$$\boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u}$$

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Objective fields

A vector field transformed according to:

$\boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u}$ is called **spatially objective vector field**.

Using this definition we can defined a spatially objective 2nd tensor filed. v

For two spatially objective vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ seen by the observer R , are related by:

$$\boldsymbol{w} = \boldsymbol{L}\boldsymbol{v}.$$

Since they are objective, the observer R^* sees

$$\boldsymbol{w}^* = \boldsymbol{Q}\boldsymbol{w} \quad \text{and} \quad \boldsymbol{w}^* = \boldsymbol{L}^*\boldsymbol{v}^*$$

$$\boldsymbol{w}^* = \boldsymbol{Q}\boldsymbol{w} = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{v} = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T\boldsymbol{v}^*$$

$$\boldsymbol{v}^* = \boldsymbol{Q}\boldsymbol{v}$$

$$\boldsymbol{L}^* = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T$$

A tensor transformed according to the last relation is **spatially objective tensor** or independent of the reference frame.

In summary

A scalar quantity ϕ is objective if and only if (iif) $\phi^* = \phi;$

A vector quantity $\boldsymbol{f}$ is **materially objective** iff $\boldsymbol{f}^* = \boldsymbol{f};$

A vector quantity $\boldsymbol{f}$ is **spatially objective** iff $\boldsymbol{f}^* = \boldsymbol{Q}\boldsymbol{f};$

A tensor quantity $\boldsymbol{T}$ is **materially objective** iff $\boldsymbol{T}^* = \boldsymbol{T};$

A tensor quantity $\boldsymbol{T}$ is **spatially objective** iff $\boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T$

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Objectivity of velocity and acceleration

We have for the velocity $\boldsymbol{V}(\boldsymbol{X},t) = \dot{\boldsymbol{\chi}}(\boldsymbol{X},t)$

and acceleration $\boldsymbol{A}(\boldsymbol{X},t) = \ddot{\boldsymbol{\chi}}(\boldsymbol{X},t)$

$$\boldsymbol{\chi}^*(\boldsymbol{X},t^*) = \boldsymbol{Q}(t)\boldsymbol{\chi}(\boldsymbol{X},t) + \boldsymbol{c}(t)$$

$$\begin{aligned}
\boldsymbol{V}^*(\boldsymbol{X},t^*) &= \boldsymbol{Q}(t)\boldsymbol{V}(\boldsymbol{X},t) + \dot{\boldsymbol{c}}(t) + \dot{\boldsymbol{Q}}(t)\boldsymbol{\chi}(\boldsymbol{X},t) \\
\boldsymbol{A}^*(\boldsymbol{X},t^*) &= \ddot{\boldsymbol{\chi}}^*(\boldsymbol{X},t^*) \\
&= \boldsymbol{Q}(t)\ddot{\boldsymbol{\chi}}(\boldsymbol{X},t) + \ddot{\boldsymbol{c}}(t) + \ddot{\boldsymbol{Q}}(t)\boldsymbol{\chi}(\boldsymbol{X},t) \\
&\quad + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{V}(X,t)\,.
\end{aligned}$$

→ The definitions of the velocity and acceleration are relative and inextricably linked to the observer.

For the deformation gradient tensor we have

$$\begin{aligned}
\boldsymbol{F}^*(\boldsymbol{X},t^*) &= \frac{\partial\boldsymbol{\chi}^*(\boldsymbol{X},t)}{\partial\boldsymbol{X}} \\
&= \frac{\partial\boldsymbol{\chi}^*(\boldsymbol{X},t)}{\partial\boldsymbol{\chi}(\boldsymbol{X},t)}\,\frac{\partial\boldsymbol{\chi}(\boldsymbol{X},t)}{\partial\boldsymbol{X}} \\
&= \boldsymbol{Q}(t)\boldsymbol{F}(\boldsymbol{X},t)
\end{aligned}$$

and

$$J^* = \det\boldsymbol{F}^*(\boldsymbol{X},t^*) = \det\boldsymbol{F}(\boldsymbol{X},t) = J$$

Starting from the definitions of the corresponding Tensors it can be shown that:

$$\boldsymbol{C}^* = \boldsymbol{C} \qquad \boldsymbol{E}^* = \boldsymbol{E}$$

$$\boldsymbol{c}^* = \boldsymbol{Q}\boldsymbol{c}\boldsymbol{Q}^T \qquad \boldsymbol{e}^* = \boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T$$